CONTINUED FRACTION EXPANSIONS IN DYNAMICAL SYSTEMS: EXAMPLES

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Abstract
Continued fractions and dynamical systems are very connected to each other, and in this article, we show how simple iterative maps can generate continued fraction expansions of real numbers. We focus on several important systems — including the Gauss Map, Farey Map, Slow Continued Fraction Map, Rényi Map, β-Transformation, and a simplified billiard model — and investigate how they uncover number-theoretic properties such as rational approximations, Diophantine analysis, and chaotic behavior. Building on the theoretical groundwork from Continued Fractions by Wieb Bosma and Cor Kraaikamp, we offer clear definitions and interpret each system from a dynamical perspective. All models are implemented in MATLAB to simulate orbits, extract continued fraction digits, and visualize invariant measures. These simulations reveal the surprisingly structured yet chaotic nature of these systems and demonstrate their value in studying the real number line. This work is for both learning and further research in number theory and dynamical systems.

Key words: Continued fractions, Dynamical systems, Gauss map, Farey map, Slow continued fraction map, Rényi map, β-transformation, Number theory, Diophantine approximation, Chaos theory, Ergodic systems, MATLAB simulations, Rational approximation, Invariant measures, Orbit analysis.

Introduction

A continued fraction represents a real number as a nested sequence of integers, written in the form:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Unlike decimal expansions, which may be long or non-repeating without revealing much structure, continued fractions often highlight deep arithmetic properties of numbers.

A dynamical system is a mathematical framework used to describe how a point evolves over time following a fixed rule. The basic process begins with an initial value x_0, applies a function T, and repeats the process iteratively:

$$x_1 = T(x_0), \quad x_2 = T(x_1), \quad x_3 = T(x_2).$$

The sequence $\{x_n\}$ is called the orbit of x_0. Even when T is a simple function, the resulting behavior can be highly complex — including phenomena such as dependence on initial conditions, also known as chaos.

By applying iterative maps to the unit interval—extracting integer parts and inverting the remaining fractions—we generate orbits that directly correspond to the continued fraction digits of the starting value. In this way, continued fraction expansions emerge as the trajectories of well-defined dynamical processes.

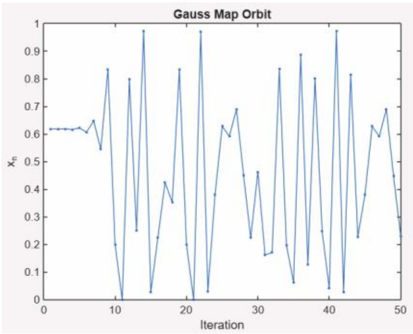
We explore that connection through five key dynamical systems:

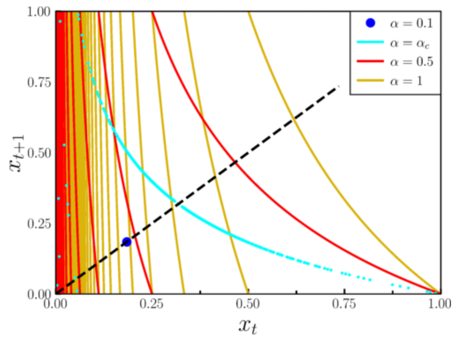
1. **The Gauss Map** – the classical model for generating regular continued fractions.
2. **The Farey Map** – rooted in Farey sequences and rational approximation theory.
3. **The Slow Continued Fraction Map** – inspired by the Euclidean algorithm.
4. **The Rényi (Backward) Continued Fraction Map** – based on ceiling functions.
5. **The β-Transformation** – a generalization with strong chaotic behavior.

Examples

1. Gauss Map – the classic map used to generate regular continued fractions. It removes the integer part and takes the reciprocal repeatedly:

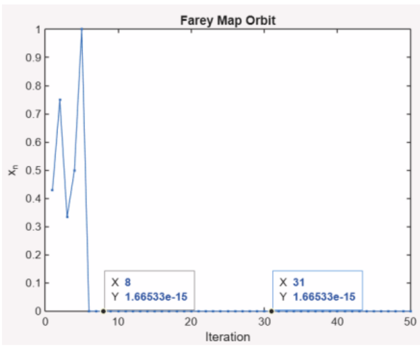
$$G(x) = \frac{1}{x} \quad \mathrm{mod}1 = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \quad x \in (0,1]$$

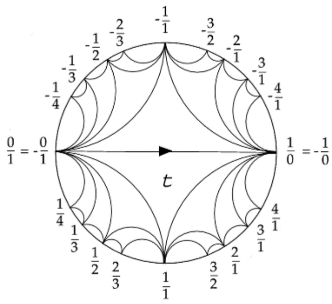




2. Farey Map – piecewise-defined and closely tied to Farey sequences. It preserves rational numbers and is used in Diophantine approximation:

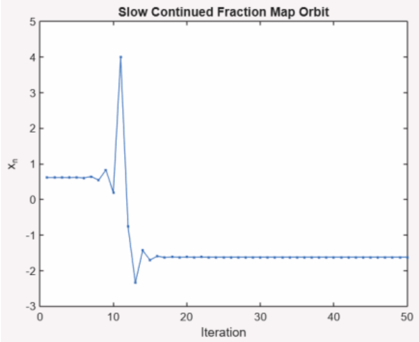
$$F(x) = \begin{cases} \frac{x}{1-x}, & 0 < x \le \frac{1}{2} \\ \frac{1-x}{x}, & \frac{1}{2} < x < 1 \end{cases}$$

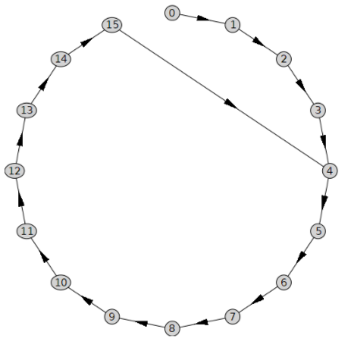




3. Slow Continued Fraction Map – converges slowly and mimics the Euclidean algorithm by subtracting before reciprocating.

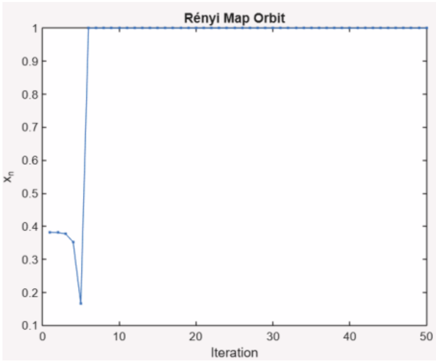
$$S(x) = \frac{1}{x} - 1, \quad x \in (0,1)$$

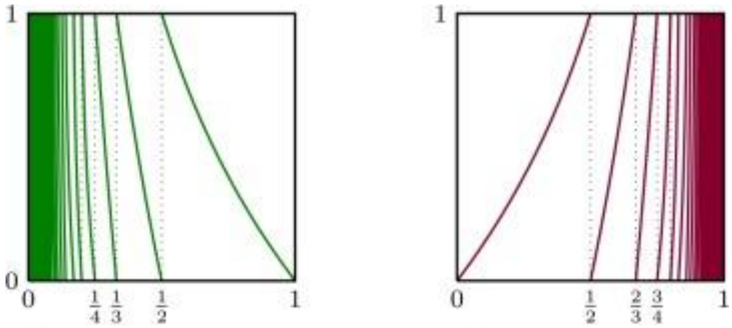




4. Renyi (Backward) Map – instead of using the floor function like Gauss, it uses the ceiling function

$$R(x) = \left\lceil \frac{1}{x} \right\rceil - \frac{1}{x}, \quad x \in (0,1)$$

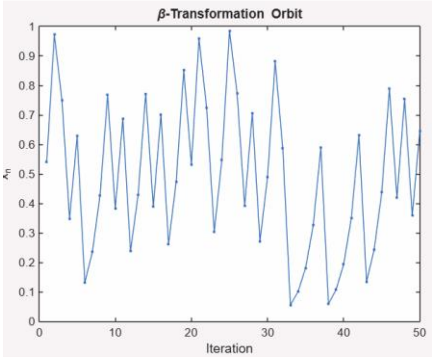


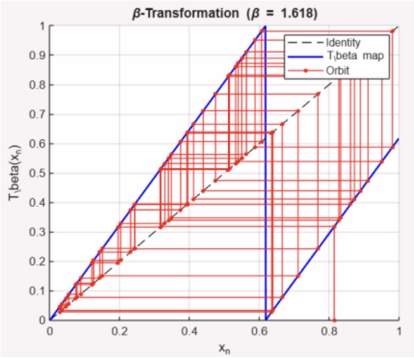


(a) The Gauss map T_0 (b) The Rényi map T_1

5. β – Transformation – this map multiplies by a base β > 1 and removes the integer part. It's chaotic and used in digit expansions for non-integer bases.

$$T_\beta(x) = \beta x \mod 1$$





Conclusion

We have shown that continued fraction expansions are not merely algebraic curiosities but arise organically from the study of dynamical systems. Through the analysis of the Gauss map and its various extensions, we have explored the deep connections between number theory and dynamical behavior.

Key Findings:

- Continued fraction digits can be generated via deterministic dynamical systems.
- Many of these systems exhibit ergodic and chaotic properties.
- MATLAB simulations provide an accessible and visual way to understand abstract mathematical behavior.

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