CPA-STRUCTURE ON SMALL-DIMENSIONAL NILPOTENT LIE ALGEBRAS

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ANNOTATION:
This work investigates commutative post-Lie algebra (CPA) structures on three-dimensional nilpotent Lie algebras. CPA-structures, a subset of post-Lie algebra structures, are essential for understanding the geometry of nil-affine actions and their connection to nil-affine crystallographic groups. These structures are also closely related to Poisson and Poisson-admissible algebras. By analyzing the bilinear product and the defining identities of CPA-structures, the research classifies and constructs such structures for specific three-dimensional nilpotent Lie algebras. Results provide insights into the algebraic properties and highlight areas for further exploration in higher-dimensional cases.

KEYWORDS: CPA-structure, nilpotent Lie algebra, post-Lie algebra, Poisson algebra, algebraic structures, commutative multiplication, nil-affine actions.

Several configurations gave rise to post-Lie algebras and post-Lie algebra structures. In order to characterize simply transitive nil-affine actions of nilpotent Lie group *G* on another nilpotent Lie group *N*, we devised these structures in [3]. This is relevant to the theory of complete nil-affinely flat manifolds and nil-affine crystallographic groups. Post-Lie algebras emerge as a natural common generalization of LR-algebras and pre-Lie algebras. On the other hand, Vallette [1] separately presented post-Lie algebras in relation to the study of Koszul operads and the homology of partition posets.

The existence of post-Lie algebra structures for a given pair of Lie algebras is an important topic coming here from geometry. In general terms, this is an extremely difficult subject, and the solution depends significantly on the algebraic characteristics of the two specified Lie algebras. See [4] for a survey on the results and unresolved problems. Commutative structures, or *CPA-structures*, provide an essential class of *post-Lie algebra structures*. In comparison with general post-Lie algebra structures, these structures are much more accessible, and we are able to provide answers to a number of concerns about their existence and classification, which helps us identify areas of study for the larger case of post-Lie algebra structures. Refer to [2] for CPA-structures on some classes filiform Lie algebras.

In this work we construct CPA-structures for three-dimensional nilpotent Lie algebras.

Definition 1. A "*commutative post-Lie algebra structure* or "CPA-structure"on a Lie algebra A is a K-bilinear product $x \cdot y$ satisfying the identities:

$$x \cdot y = y \cdot x \qquad (1)$$

$$[x,y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z) \qquad (2)$$

$$x \cdot [y,z] = [x \cdot y,z] + [y,x \cdot z] \qquad (3)$$

for all $x,y,z \in A$.

It turns out that certain CPA-structures are related to Poisson algebras and Poissonadmissible algebras.

Consider following three-dimensional algebra with multiplication

$\mathfrak{h}$: $[e_1,e_2] = e_3$,

$(L(\mathfrak{g}_3),[-,-])$:$[e_1,e_2] = e_1$, $[e_2,e_3] = e_2$ (omitted products are zero).

$\mathfrak{g}_2^{\alpha}$: $[e_1,e_3] = e_1+ e_2$, $[e_2,e_3] = \alpha e_2$.

$\mathfrak{sl}_2$: $[e_1,e_2] = e_1$, $[e_1,e_3] = -e_2$, $[e_2,e_3] = e_1$.

Preposition 1. There are CPA-structure for 3-dimensional nilpotent Lie algebras with

non-trivial commutative $\cdot$ multiplication:

$(L(\mathfrak{h}),[-,-],\cdot)$	$[e_1,e_2] = e_3$	$e_1 \cdot e_1 = e_2$
$(L(\mathfrak{g}_3),[-,-],\cdot)$:	$[e_1,e_2] = e_1$, $[e_2,e_3] = e_2$	$e_3 \cdot e_3 = ve_1 + we_2$;
$(L(\mathfrak{g}_3),[-,-],\cdot)$:	$[e_1,e_2] = e_1$, $[e_2,e_3] = e_2$	$e_3 \cdot e_3 = ve_1 + we_2$;
$\mathfrak{g}_2^{\alpha}$	$[e_1,e_3] = e_1+ e_2$, $[e_2,e_3] = \alpha e_2$.	
		(omitted products are zero).

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